The Howard County Public School System

Secondary Mathematics Lesson Plan

Course: AP Calculus C/GT
Instructor: Michelle Krummel

Quarter I **MSDE Standard** ________

HCPSS Unit III **HCPSS Goal** ________

Lesson Title: Infinite Series

Objective(s): The student will be able to use a problem-solving approach to find the sums of convergent infinite series, and use proper sigma notation. The student will be able to represent an infinite series geometrically.

Duration: This lesson will be completed over three 90-minute class sessions. During the first sessions, students will explore patterns of infinite series represented geometrically. During the second and third sessions students will complete an original piece of artwork representing an infinite series.

Materials:

☐ Manipulatives

☐ Measurement Tools

☐ Scientific / Graphing Calculator

☒ Computer Software

- http://demonstrations.wolfram.com
- Microsoft Paint (or other image editing software)

☒ Other (list below)

- Worksheet: Infinite Series Investigation
- PowerPoint Presentation: Infinite Series in Art
- Colored pencils, pastels, magazine clippings, colored paper, fabric, tissue paper, watercolors, paint, etc.
- Art paper, poster board, scissors, glue

Warm-Up/Pre-Assessment:

1. Demonstrate the construction of a Triangular Baravelle Spiral using the Mathematica Player found at http://demonstrations.wolfram.com.
2. Construct a Square Baravelle Spiral using the Mathematica Player

Teaching Strategies

☒ Higher-Order Questioning
☒ Technology Integration
☐ Data Analysis
☐ Independent Reading
☒ Interpretation of Graphics
☐ Estimation
☐ Extension
☐ Concept Attainment
☐ Think-Pair-Share
☐ Roundtable
☐ Jigsaw
☐ Pairs Check/Review
☒ Independent/Group Project
☐ Interactive Student Notebook®
☐ Writing
☐ Modeling Demonstration
☐ Think Aloud
☐ Reciprocal Teaching
☐ Group Activities
☐ Simulation
☐ Use of Video Clip
☒ Cross-Curricular Connections
☐ Other (explain):

Differentiation

☒ Content
☐ Process
☐ Product

Learning Modalities

☒ Visual
☐ Auditory
☒ Tactile/Kinesthetic

Accommodations

☐ Extended time
☐ Preferential seating
☐ Learning partner
☐ Alternative reading
☐ Guided notes
☐ Framed paragraphs
☐ Other (explain):

found at http://demonstrations.wolfram.com. Using Microsoft Paint or paper and colored pencils, ask students to identify and color the segments that make up a single spiral. Then identify the total number of spirals found in the figure.

Development/Procedures:

1. Using whole group discussion, find a recursive formula for the area of a single spiral in the Square Baravelle Spiral design. Represent this area as an infinite geometric series, and apply the formula for finding the sum of an infinite geometric series to compute the area.

2. In cooperative groups, examine the construction of Sierpinski's Triangle. Find a formula for the area and perimeter of Sierpinski's Triangle. Relate these formulas to infinite series.

3. In cooperative groups, examine the construction of the Koch Snowflake. Find a formula for the area and perimeter of the Koch Snowflake. Relate these formulas to infinite series.

    EXTENSION: The Sierpinski Triangle and Koch Snowflake are examples of fractals. Students who are interested in further investigation of this topic may research other examples of fractals that are related to infinite series.

4. In cooperative groups, examine the construction of Gabriel's Wedding Cake. Find a formula for the volume and surface area of the wedding cake. Relate these formulas to infinite series.

    DIFFERENTIATION: More advanced students may investigate the volume and surface area of Gabriel's Horn (a more complex figure than Gabriel's Wedding Cake).

5. Show PowerPoint Presentation with samples of artwork depicting infinite series. Discuss the use of complimentary and analogous colors used in the samples. Discuss the use of warm versus cool colors in the samples.

6. Assign each student the task of creating a unique work of art depicting an infinite series. Artwork must make use of either analogous or complimentary colors. Students may work with the medium of their choice (colored pencil, pastels, paint, watercolor, fabric, torn paper, magazine cutouts, etc.). Students who choose to do a collage should have a unifying theme.

Reading Strategies

- ☐ Application
- ☐ Prior knowledge
- ☐ Preview
- ☐ Voc./Concepts
- ☐ Self-monitoring through clarifying questions
- ☐ Reread
- ☐ Summarize or paraphrase
- ☐ Uses rubrics
- ☐ Vocabulary
- ☐ Comprehension
- ☐ Other (explain):

Assessment

- ☐ Collect and Grade
- ☐ Check for Completion
- ☐ In-Class Check
- ☒ Rubric
- ☒ Peer / Self Assessment
- ☐ Journal / Learning Log
- ☐ Portfolio
- ☐ Constructed Response
- ☐ Quiz
- ☐ Test
- ☐ Presentation
- ☒ Performance Assessment
- ☐ Informal Assessment
- ☐ Exit Slip
- ☐ Other (explain):

Closure:

1. How can infinite series be represented geometrically? Give an example.

2. What paradoxes were found when examining the area and perimeter (or volume and surface area) of the figures studied?

Assessment:

1. Write the formulas for finding the area and perimeter of the Sierpinski Triangles and the Koch Snowflakes. Write formulas for finding the volume and surface are of Gabriel's Wedding Cake.
2. Evaluate student artwork. Artwork must represent an infinite series in some way. Artwork must also make use of either complimentary or analogous colors.
3. Students complete the following self-reflection questions:.
    a. In what ways did you experiment mathematically while working on this assignment?
    b. In what ways did you experiment artistically while working on this assignment?
    c. What problems did you encounter while working on this assignment? How did you solve those problems?
    d. In what way is your artwork related to infinite series? Be specific.
    e. Did you enjoy working on this assignment? Why or why not?

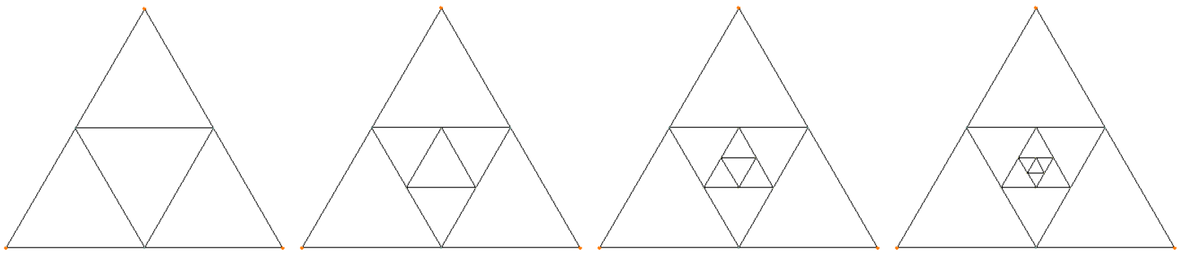
Name: ______________________________
Date: ___________________ Pd: _______

Infinite Series Investigation

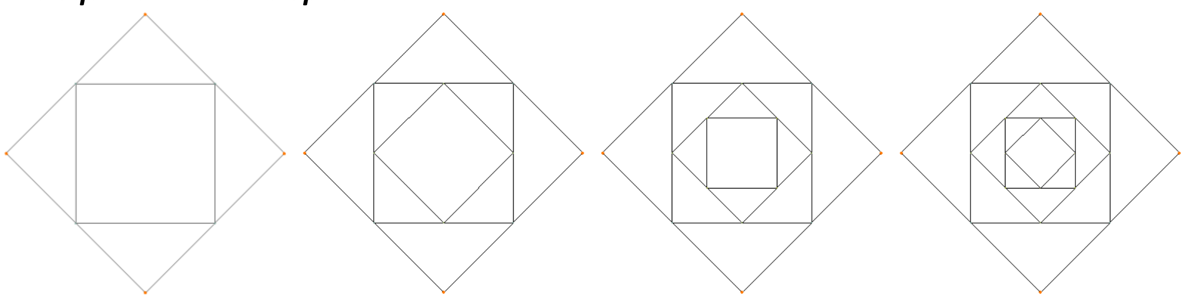
Baravelle Spirals

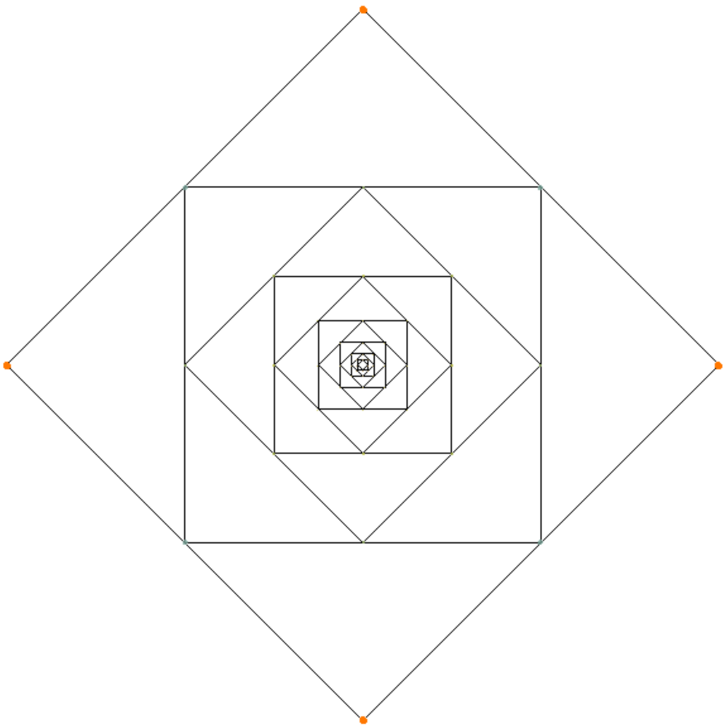
Baravelle Spirals are formed using regular polygons. To construct a Baravelle Spiral, begin with a regular *n*-sided polygon. At each iteration (stage *n*), connect the midpoints of the sides of the polygon to create another, smaller polygon.

The Triangular Baravelle Spiral



The Square Baravelle Spiral





1. Color the figure, using four different colors, to show the spirals. Make a conjecture regarding the area of a spiral in the figure.

Find a formula for the area of a spiral at stage *n*. Use the formula to find the total area of the spiral.

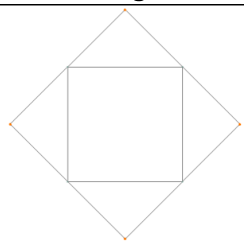
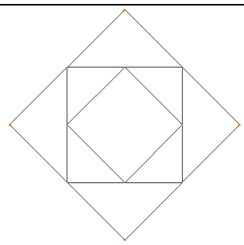
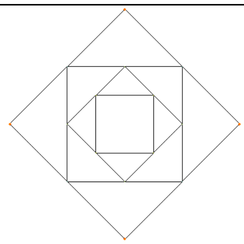
Figure	**Area**
 n=0	$A_0 =$
 n=1	$A_1 =$
 n=2	$A_2 =$
Stage *n*	$A_n =$
Each Spiral	

Table 1: Determining Area of a Baravelle Spiral

2. In what way is this problem related to infinite series?

Some Special Infinite Series

You will be working with three special types of infinite series: the geometric series, the harmonic series, and the p-series.

- An infinite geometric series is a series of the form $\sum_{n=0}^{\infty} ar^n$. This series converges when $|r| < 1$.
- The harmonic series is the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}$. This series diverges.
- A p-series is a series of the form $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p$. The p-series converges when $p>1$. A special case of the p-series, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^2$, where $p=2$, converges to $\frac{\pi^2}{6}$.

Sierpinski's Triangle

Sierpinski's Triangle is a well-known fractal. To construct Sierpinski's Triangle, begin with a filled equilateral triangle (stage 0). At each iteration (stage *n*), remove the equilateral triangle formed by connecting the midpoints of the sides of each remaining filled triangle.

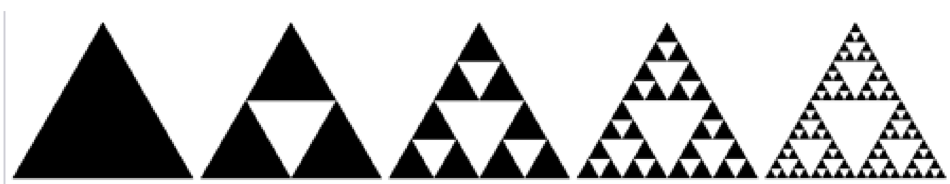


Figure 1: Constructing the Sierpinski Triangle

3. Make a conjecture regarding the area and perimeter of the figure.

Find a formula for the area and perimeter of the Sierpinski Triangle at stage *n* and record it in the table below. Use the formula to calculate the total area and total perimeter of the Sierpinski Triangle.

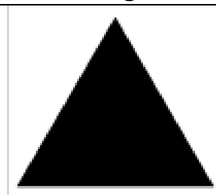
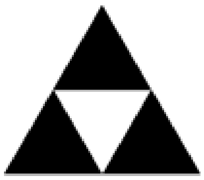
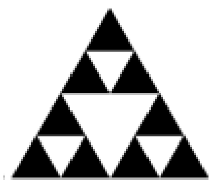
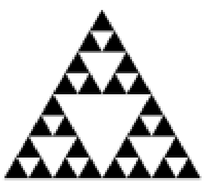
Figure	Area	Perimeter
 n=0	$A_0 =$	$P_0 =$
 n=1	$A_1 =$	$P_1 =$
 n=2	$A_2 =$	$P_2 =$
 n=3	$A_3 =$	$P_3 =$
Stage *n*	$A_n =$	$P_n =$
Sierpenski Triangle		

Table 2: Determining Area and Perimeter of the Sierpenski Triangle

4. In what way is this problem related to infinite series?

5. In what way do your findings represent a paradox?

The Koch Snowflake

The Koch Snowflake is another well-known fractal. To construct the snowflake, start with an equilateral triangle (three faces). Each face is a line segment. To move to the next stage, divide each line segment into thirds and construct an equilateral triangle using the middle third as the base. To complete the stage, remove the base of the newly added triangle (see Figure 2).

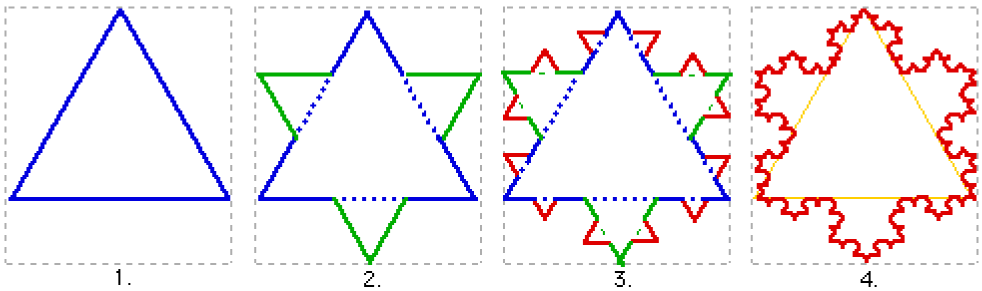
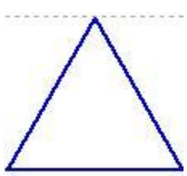
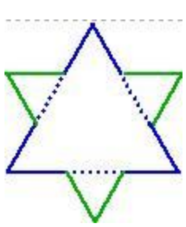


Figure 2: Koch Snowflake Construction

6. Make a conjecture about the number of faces, the area, and the perimeter of the Koch Snowflake.

Find a formula for the number of faces, the area, and the perimeter of the Koch Snowflake at stage *n*. Use the formula to find the total number of faces, total area, and total perimeter of the Koch Snowflake.

Figure	Num. Faces	Area	Perimeter
		$A_0 =$	$P_0 =$
		$A_1 =$	$P_1 =$

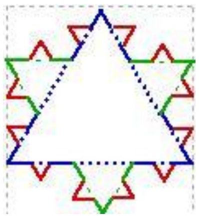
		$A_2 =$	$P_2 =$
Stage *n*		$A_n =$	$P_n =$
Koch Snowflake			

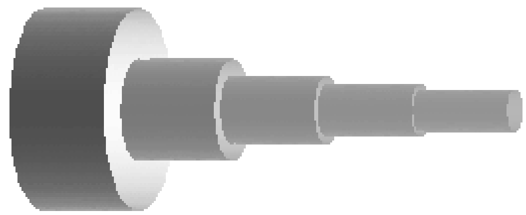
Table 3: Area and Perimeter of the Koch Snowflake

7. In what way do your findings represent a paradox?

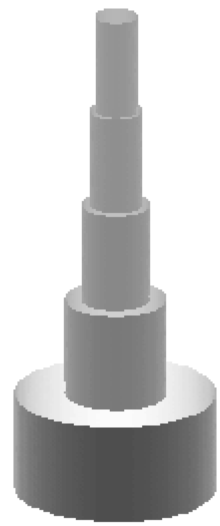
Gabriel's Wedding Cake

We construct Gabriel's wedding cake by revolving the graph of a step function about the x-axis:

$$f(x)=\begin{cases} 1, & \text{if } 1\le x<2 \\ 1/2, & \text{if } 2\le \text{x} <3 \\ \vdots & \\ 1/n, & \text{if } n\le x<n+1 \\ \vdots & \end{cases}$$



Each layer of the cake is a cylinder. We determine the volume of the wedding cake by summing the volumes of the cylinders. Write a formula for finding the volume of the wedding cake using summation notation.

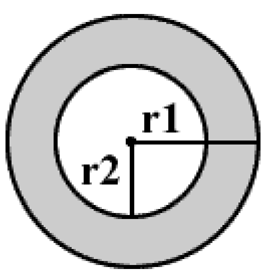


$V=$

Use Euler's p-series formula for p = 2 (refer to page 1) to find the volume.

$V=$

Determine the surface area by considering the tops and sides of the layers separately. **The top of each layer** forms an annulus (a ring-shaped figure). Write the formula for finding the area of the n^{th} annulus:



Area of the n^{th} annulus =

Notice that the areas of the tops summed together yield a telescoping series. Write the formula for finding the total area of the tops, then find the sum.

Area of the tops =

Write the formula for finding **the lateral area** (area of the sides) of the n^{th} layer:

Area of the side of the n^{th} layer = (circumference)(height) =

Notice that the areas of the sides summed together yield a multiple of the harmonic series. Write the formula for finding the total area of the sides, then find the sum (refer to page 1).

Total area of the sides =

Thus we see that one can make enough dough to bake Gabriel's cake, but cannot make enough frosting to cover it!

References

1. Tim Howard, Columbus State University, web site at http://math.colstate.edu/thoward/
2. Michael Barnsley, *Fractals Everywhere*, Academic Press, Boston (1988).
3. Bellevue Community College, "The Snowflake Curve" web site at http://scidiv.bcc.ctc.edu/Math/Snowflake.html.
4. Robert Devaney, "Chaos in the Classroom" web site at http://math.bu.edu/DYSYS/chaos-game/chaos-game.html.
5. Julian Fleron, "Gabriel's Wedding Cake," *The College Mathematics Journal* (30), 1: 35-38 (1999).
6. R. Larson, R. Hostetler, and B. Edwards, "Exercise 74: Sphereflake", *Calculus with Analytic Geometry, Sixth Edition*, pp. 545 and 566, Houghton Mifflin Company, Boston (1998).
7. R. Young, "Summing the Reciprocals of the Squares", appearing in *Excursions in Calculus*, pp. 338-56, Mathematical Association of America (1992).